Some Results On Point Visibility Graphs *

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Abstract

In this paper, we present two necessary conditions for recognizing point visibility graphs and conjecture that they are sufficient. We show that this recognition problem lies in PSPACE. In addition, we state several properties of point visibility graphs. For planar point visibility graphs, we present a complete characterization which leads to a linear time recognition and reconstruction algorithm.

1 Introduction

The visibility graph is a fundamental structure studied in the field of computational geometry and geometric graph theory [3, 7]. Some of the early applications of visibility graphs included computing Euclidean shortest paths in the presence of obstacles [10] and decomposing two-dimensional shapes into clusters [13]. Here, we consider problems from visibility graph theory.

Let $P = \{p_1, p_2, ..., p_n\}$ be a set of points in the plane (see Figure 1). We say that two points p_i and p_j of P are *mutually visible* if the line segment p_ip_j does not contain or pass through any other point of P. In other words, p_i and p_j are visible if $P \cap p_ip_j = \{p_i, p_j\}$. If two vertices are not visible, they are called an *invisible pair*. For example, in Figure 1(c), p_1 and p_5 form a visible pair whereas p_1 and p_3 form an invisible pair. If a point $p_k \in P$ lies on the segment p_ip_j connecting two points p_i and p_j in P, we say that p_k blocks the visibility between p_i and p_j, and p_k is called a *blocker* in P. For example in Figure 1(c), p_5 blocks the visibility between p_1 and p_3 as p_5 lies on the segment p_1p_3.

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The *visibility graph* (also called the *point visibility graph* (PVG)) G of P is defined by associating a vertex v_i with each point p_i of P such that (v_i, v_j) is an undirected edge of G if p_i and p_j are mutually visible (see Figure 1(a)). Observe that if no three points of P are collinear, then G is a complete graph as each pair of points in P is visible since there is no blocker in P. Sometimes the visibility graph is drawn directly on the point set, as shown in Figures 1(b) and 1(c), which is referred to as a *visibility embedding* of G.

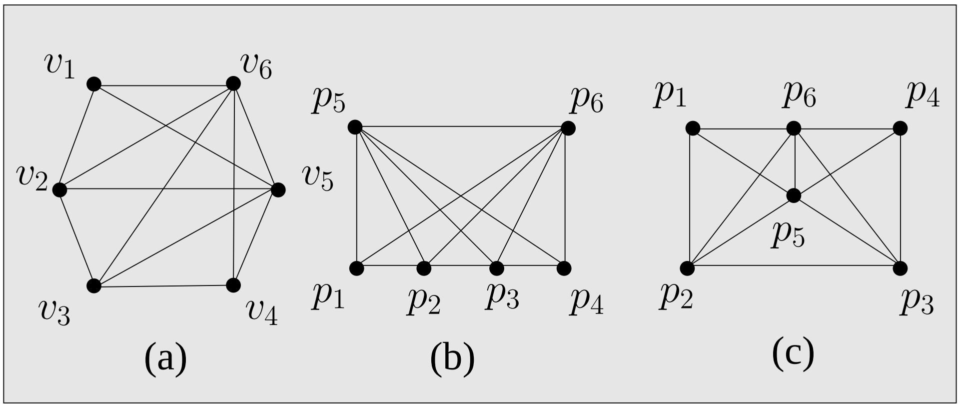

Figure 1: (a) A point visibility graph with (v_1, v_2, v_3, v_4) as a CSP. (b) A visibility embedding of the point visibility graph where (p_1, p_2, p_3, p_4) is a GSP. (c) A visibility embedding of the point visibility graph where (p_1, p_2, p_3, p_4) is not a GSP.

Given a point set P, the visibility graph G of P can be computed as follows. For each point p_i of P, the points of P are sorted in angular order around p_i. If two points p_j and p_k are consecutive in the sorted order, check whether p_i, p_j and p_k are collinear points. By traversing the sorted order, all points of P, that are not visible from p_i, can be identified in $O(n \log n)$ time. Hence, G can be computed from P in $O(n^2 \log n)$ time. Using the result of Chazelle et al. [2] or Edelsbrunner et al. [5], the time complexity of the algorithm can be improved to $O(n^2)$ by computing sorted angular orders for all points together in $O(n^2)$ time.

Consider the opposite problem of determining if there is a set of points P whose visibility graph is the given graph G. This problem is called the visibility graph *recognition* problem. Identifying the set of properties satisfied by all visibility graphs is called the visibility graph *characterization* problem. The problem of actually drawing one such set of points P whose visibility

graph is the given graph G, is called the visibility graph *reconstruction* problem.

Here we consider the recognition problem: Given a graph G in adjacency matrix form, determine whether G is the visibility graph of a set of points P in the plane [8]. We present two necessary conditions for this recognition problem in Section 2 along with some properties of point visibility graphs, and conjecture that they are sufficient. Though the first necessary condition can be tested in $O(n^3)$ time, it is not clear whether the second necessary condition can be tested in polynomial time. On the other hand, we show in Section 3 that the recognition problem lies in PSPACE.

If a given graph G is planar, there can be two cases: (i) G has a planar visibility embedding, and (ii) G does not have any planar visibility embedding. The former case has been characterized by Eppstein [4] by presenting four infinite families of G. We characterize the latter case by adding two more infinite families of G. Using these characterizations we present in Section 4 an $O(n)$ algorithm for recognizing and reconstructing G. Note that this algorithm does not require any prior embedding of G. Finally, we conclude the paper with a few remarks.

2 Properties of point visibility graphs

Consider a subset S of vertices of G such that their corresponding points C in a visibility embedding of G are collinear. The path formed by the points of C is called a *geometric straight path* (GSP). For example, the path (p_1, p_2, p_3, p_4) in Figure 1(b) is a GSP as the points p_1, p_2, p_3 and p_4 are collinear. Note that there may be another visibility embedding of G as shown in Figure 1(c), where points p_1, p_2, p_3 and p_4 are not collinear. So, the points forming a GSP in a visibility embedding of G may not form a GSP in every visibility embedding of G. If a GSP is a maximal set of collinear points, it is called a *maximal geometric straight path* (max GSP). In the following, we state some properties of PVGs and present two necessary conditions for recognizing G.

Lemma 1 *If G is a PVG but not a path, then for any GSP in any visibility embedding of G, there is a point visible from all the points of the GSP[9].*

Proof: For every GSP, there exists a point p_i whose perpendicular distance to the line containing the GSP is the smallest. So, all points of the GSP are visible from p_i. □

Let H be a path in G such that no edges exist between any two non-consecutive vertices in H. We call H a *combinatorial straight path* (CSP). Observe that in a visibility embedding of G, H may not always correspond to a GSP. In Figure 1(a), $H = (v_1, v_2, v_3, v_4)$ is a CSP which corresponds to a GSP in Figure 1(b) but not in Figure 1(c). Note that CSP always refers to a path in G, whereas GSP refers to a path in a visibility embedding of G. A CSP that is a maximal path without back edges, is called a *maximal combinatorial straight path* (*max* CSP).

Lemma 2 *If a vertex v_i does not belong to a max CSP in G, then the degree of v_i is at least the number of vertices in the max CSP.*

Proof: Let $(p_j, p_{j+1}, ..., p_m)$ be the corresponding GSP of the max CSP. If p_i is visible from every point of the GSP, then the property holds. Otherwise, if p_i is not visible from any point p_l on the GSP, then there is a blocker on the segment (p_i, p_j) that is visible from p_i. Thus, the degree of v_i is at least the size of the max CSP. □

Lemma 3 *G is a PVG and bipartite if and only if the entire G is a CSP.*

Proof: If the entire G can be embedded as a GSP, then alternate points in the GSP form the bipartition and the lemma holds. Otherwise, there exists at least one max GSP which does not contain all the points. By Lemma 1, there exists one point p_i adjacent to all points of the GSP. So, p_i must belong to one partition and all points of the GSP (having edges) belong to the other partition. Hence, G cannot be a bipartite graph, a contradiction. The other direction of the proof is trivial. □

Lemma 4 *If G is a PVG, then the size of the maximum clique in G is bounded by twice the minimum degree of G, and the bound is tight.*

Proof: In a visibility embedding of G, draw rays from a point p_i of minimum degree through every visible point of p_i. Observe that any ray may contain several points not visible from p_i. Since any clique can have at most two points from the same ray, the size of the clique is at most twice the number of rays, which gives twice the minimum degree of G. □

Lemma 5 *If G is a PVG and it has more than one max CSP, then the diameter of G is 2 [9].*

Proof: If two vertices v_i and v_j are not adjacent in G, then they belong to a CSP L of length at least two. By Lemma 1, there must be some vertex v_k that is adjacent to every vertex in L. (v_i, v_k, v_j) is the required path of length 2. Therefore, the diameter of G cannot be more than two. □

Corollary 1 *If G is a PVG but not a path, then the BFS tree of G rooted at any vertex v_i of G has at most three levels consisting of v_i in the first level, the neighbours of v_i in G in the second level, and the rest of the vertices of G in the third level.*

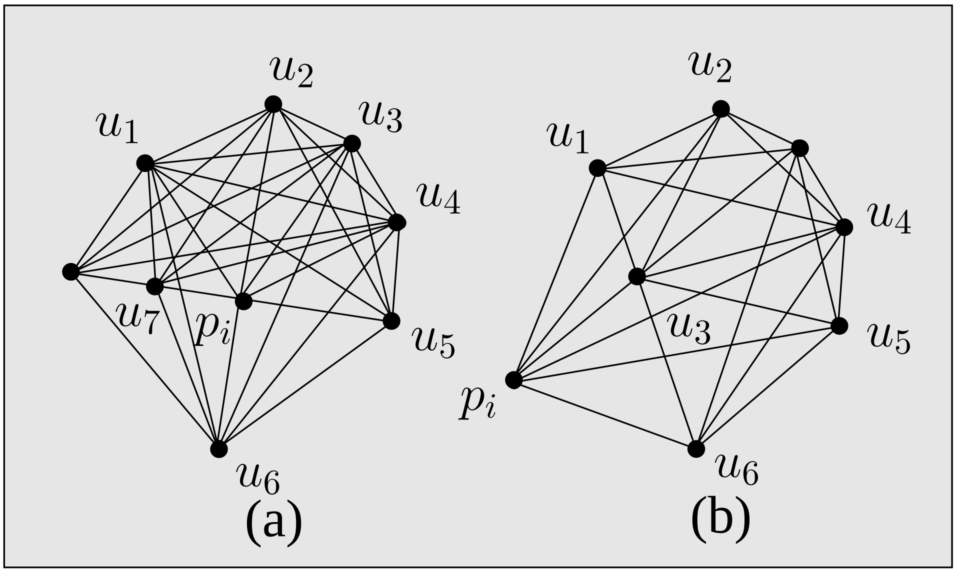

Figure 2: (a) The points $(u_1, u_2, ..., u_7,\ u_1)$ are visible from an internal point p_i. (b) The points $(u_1, u_2, ..., u_6)$ are visible from a convex hull point p_i.

Lemma 6 *If G is a PVG but not a path, then the subgraph induced by the neighbours of any vertex v_i, excluding v_i, is connected.*

Proof: Consider a visibility embedding of G where G is not a path. Let $(u_1, u_2, ..., u_k, u_1)$ be the visible points of p_i in clockwise angular order. If p_i is not a convex hull point, then $(u_1, u_2), (u_2, u_3), ..., (u_{k-1}, u_k), (u_k, u_1)$ are visible pairs (Figure 2(a)). If p_i, u_1 and u_k are convex hull points, then $(u_1, u_2), (u_2, u_3), ..., (u_{k-1}, u_k)$ are visible pairs (Figure 2(b)). Since there exists a path between every pair of points in $(u_1, u_2, ..., u_k, u_1)$, the subgraph induced by the neighbours of v_i is connected. □

Necessary Condition 1 *If G is not a CSP, then the BFS tree of G rooted at any vertex can have at most three levels, and the induced subgraph formed by the vertices in the second level must be connected.*

Proof: Follows from Corollary 1 and Lemma 6. $\square$

Let $(v_1, v_2, ..., v_k)$ be a path in G such that no two non-consecutive vertices are connected by an edge in G (Figure 3(a)). For any vertex v_j, $2 \leq j \leq k-1$, v_j is called a *vertex-blocker* of (v_{j-1}, v_{j+1}) as (v_{j-1}, v_{j+1}) is not an edge in G and both (v_{j-1}, v_j) and (v_j, v_{j+1}) are edges in G. In the same way, consecutive vertex-blockers on such a path are also called *vertex-blockers*. For example, $v_m * v_{m+1}$ is a vertex-blocker of (v_{m-1}, v_{m+2}) for $2 \leq m \leq k-2$.

Consider the graph in Figure 3(b). Though G satisfies Necessary Condition 1, it is not a PVG because it does not admit a visibility embedding. It can be seen that this graph without the edge (v_2, v_4) admits a visibility embedding (see Figure3(a)), where $(v_1, v_2, v_3, v_4, v_5)$ forms a GSP. However, (v_2, v_4) demands visibility between two non-consecutive collinear blockers which cannot be realized in any visibility embedding.

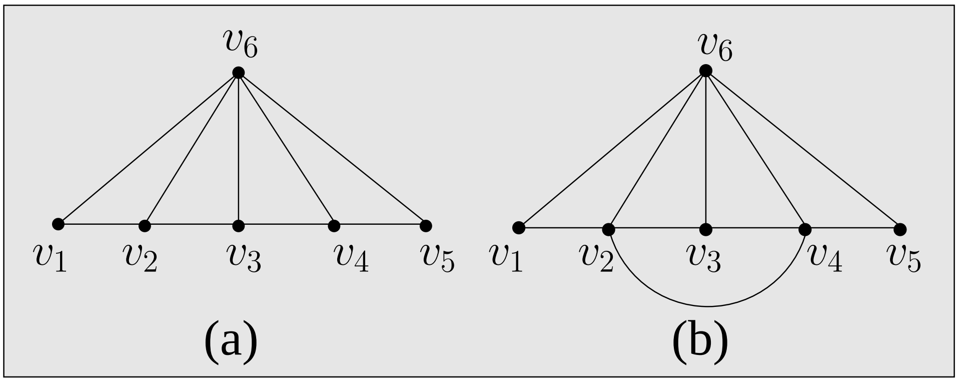

Figure 3: (a) Vertices v_2, v_3, v_4 are vertex-blockers of (v_1, v_3), (v_3, v_4) (v_3, v_5) respectively. Also, $v_2 * v_3 * v_4$ is the vertex-blocker of (v_1, v_5). (b) The graph satisfies Necessary Condition 1 but is not a PVG because of the edge (v_2, v_4).

Necessary Condition 2 *There exists an assignment of vertex-blockers to invisible pairs in G such that:*

1. *Every invisible pair is assigned one vertex-blocker.*

2. *If two invisible pairs in G sharing a vertex v_i (say, (v_i, v_j) and (v_i, v_k)), and their assigned vertex-blockers are not disjoint, then all vertices*

in the two assigned vertex-blockers along with vertices v_i, v_j and v_k must be a CSP in G.

Proof: In a visibility embedding of G, every segment connecting two points, that are not mutually visible, must pass through another point or a set of collinear points, and they correspond to vertex-blockers in G.

Since (v_i, v_j) and (v_i, v_k) are invisible pairs, the segments (p_i, p_j) and (p_i, p_k) must contain points. If there exists a point p_m on both $p_i p_j$ and $p_i p_k$, then points p_i, p_m, p_j, p_k must be collinear. So, v_i, v_m, v_j and v_k must belong to a CSP. □

Conjecture 1 *A graph G satisfying Necessary Conditions 1 and 2 is a point visibility graph.*

Lemma 7 *If the size of the longest GSP in some visibility embedding of a graph G with n vertices is k, then the degree of each vertex in G is at least* $\lceil \frac{n-1}{k-1} \rceil$ *[11, 12].*

Proof: For any point p_i in a visibility embedding of G, the degree of p_i is the number of points visible from p_i which are in angular order around p_i. Since the longest GSP is of size k, a ray from p_i through any visible point of p_i can contain at most k-1 points excluding p_i. So there must be at least $\lceil \frac{n-1}{k-1} \rceil$ such rays, which gives the degree of p_i. □

Theorem 1 *If G is a PVG but not a path, then G has a Hamiltonian cycle.*

Proof: Let $H_1, H_2, ..., H_k$ be the convex layers of points in a visibility embedding of G, where H_1 and H_k are the outermost and innermost layers respectively. Let $p_i p_j$ be an edge of H_1, where p_j is the next clockwise point of p_i on H_1 (Figure 4(a)). Draw the left tangent of p_i to H_2 meeting H_2 at a point p_l such that the entire H_1 is to the left of the ray starting from p_i through p_l. Similarly, draw the left tangent from p_j to H_2 meeting H_2 at a point p_m. If $p_l = p_m$ then take the next clockwise point of p_l in H_2 and call it p_t. Remove the edges $p_i p_j$ and $p_l p_t$, and add the edges $p_i p_l$ and $p_j p_t$ (Figure 4(a)). Consider the other situation where $p_l \neq p_m$. If $p_l p_m$ is an edge, then remove the edges $p_i p_j$ and $p_l p_m$, and add the edges $p_i p_l$ and $p_j p_m$ (Figure 4(b)). If $p_l p_m$ is not an edge of H_2, take the next counterclockwise point of p_m on H_2 and call it p_q. Remove the edges $p_i p_j$ and $p_q p_m$, and add the edges $p_i p_q$ and $p_j p_m$ (Figure 5(a)).

Thus, H_1 and H_2 are connected forming a cycle $C_{1,2}$. Without the loss of generality, we assume that $p_m \in H_2$ is the next counter-clockwise point of

p_j in $C_{1,2}$ (Figure 5(b)). Starting from p_m, repeat the same construction to connect $C_{1,2}$ with H_3 forming $C_{1,3}$. Repeat till all layers are connected to form a Hamiltonian cycle $C_{1,k}$. Note that if H_k is just a path (Figure 5(b)), it can be connected trivially to form $C_{1,k}$. □

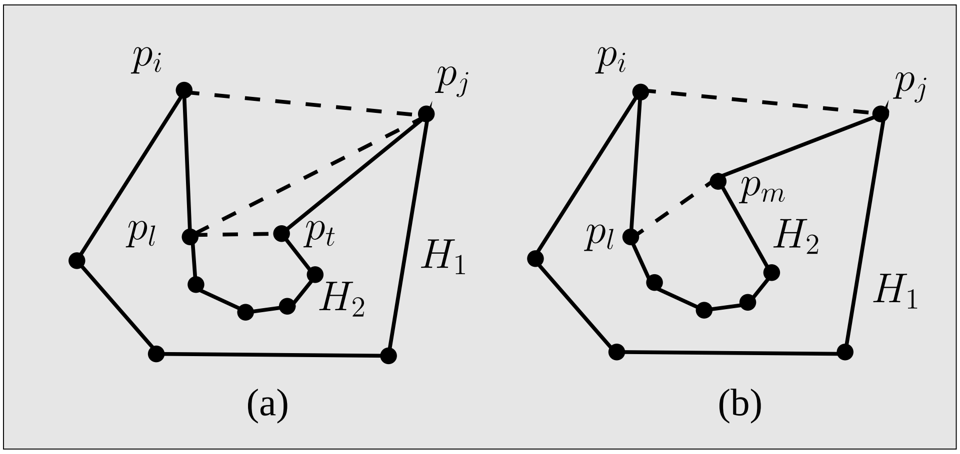

Figure 4: (a) The left tangents of p_i and p_j meet H_2 at the same point p_l. (b) The left tangents of p_i and p_j meet H_2 at points p_l and p_m of the same edge.

Corollary 2 *Given G and a visibility embedding of G, a Hamiltonian cycle in G can be constructed in linear time.*

Proof: This is because the combinatorial representation of G contains all its edges, and hence the gift-wrapping algorithm for finding the convex layers of a point set becomes linear in the input size.

Lemma 8 *Consider a visibility embedding of G. Let A, B and C be three nonempty, disjoint sets of points in it such that $\forall p_i \in A$ and $\forall p_j \in C$, the GSP between p_i and p_j contains at least one point from B, and no other point from A or C (Figure 8(a)). Then $|B| \geq |A| + |C| - 1$ [11, 12].*

Proof: Draw rays from a point $p_i \in A$ through every point of C (Figure 8(b)). These rays partition the plane into $|C|$ wedges. Since points of C are not visible from p_i, there is at least one blocker lying on each ray between p_i and the point of C on the ray. So, there are at least $|C|$ number of such blockers. Consider the remaining $|A - 1|$ points of A lying in different wedges. Consider a wedge bounded by two rays drawn through $p_k \in C$ and $p_l \in C$. Consider the segments from p_k to all points of A in the wedge.

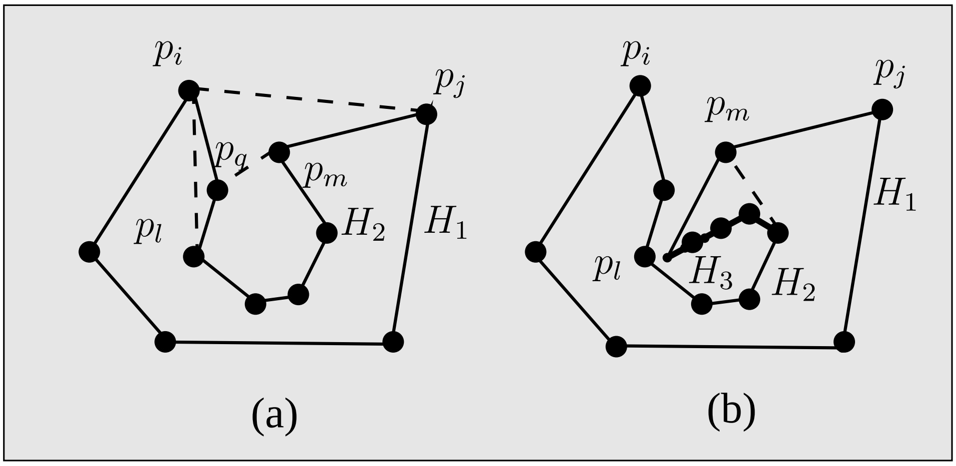

Figure 5: (a) The left tangents of p_i and p_j meet H_2 points p_l and p_m of different edges. (b) The innermost convex layer is a path which is connected to $C_{1,2}$.

Since these segments meet only at p_k, and p_k is not visible from any point of A in the wedge, each of these segments must contain a distinct blocker. So, there are at least $|A| - 1$ blockers in all the wedges. Therefore the total number of points in B is at least $|A| + |C| - 1$. □

Lemma 9 *Consider a visibility embedding of G. Let A and C be two nonempty and disjoint sets of points such that no point of A is visible from any point of C. Let B be the set of points (or blockers) on the segment p_ip_j, $\forall p_i \in A$ and $\forall p_j \in C$, and blockers in B are allowed to be points of A or C. Then $|B| \geq |A| + |C| - 1$ [12].*

Proof: Draw rays from a point $p_i \in A$ through every point of C. These rays partition the plane into at most $|C|$ wedges. Consider a wedge bounded by two rays drawn through $p_k \in C$ and $p_l \in C$. Since these rays may contain other points of A and C, all points between p_i and the farthest point from p_i on a ray, are blockers in B. Observe that all these blockers except one may be from A or C. Thus, excluding p_i, B has at least as many points as from A and C on the ray. Consider the points of A inside the wedge. Draw segments from p_k to all points of A in the wedge. Since these segments may contain multiple points from A, all points on a segment between p_k and the farthest point from p_k are blockers in B. All these points except one may be from A. Thus, B has at least as many points as from A inside the wedge.Therefore the total number of points in B is at least $|A| + |C| - 1$. □

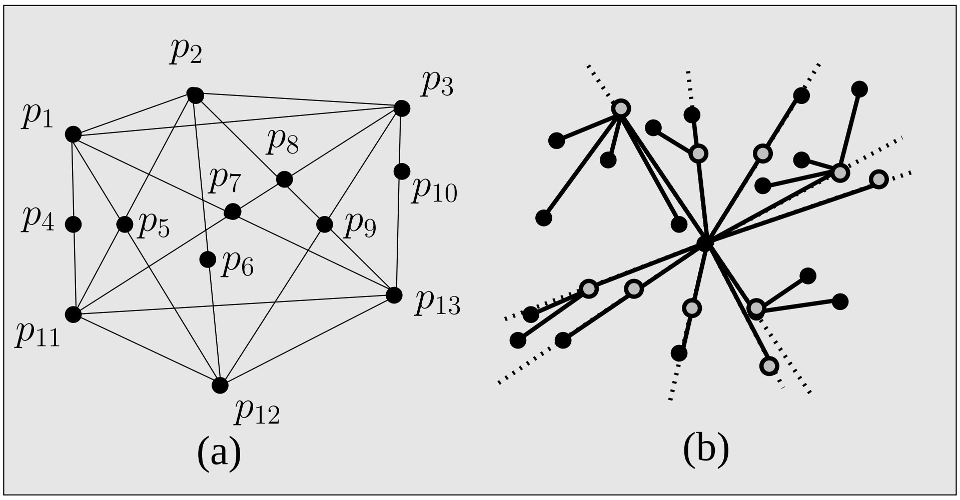

Figure 6: (a) A PVG with A = $\{p_1,\ p_2,\ p_3\}$, B=$\{p_4,\ p_5,\ p_6,\ p_7,\ p_8,\ p_9,\ p_{10}\}$ and C=$\{p_{11},\ p_{12},\ p_{13}\}$. (b) Points of A and C connected by edges representing blockers.

3 Computational complexity of the recognition problem

In this section we show that the recognition problem for a PVG lies in PSPACE. Our technique in the proof follows a similar technique used by Everett [6] for showing that the recognition problem for polygonal visibility is in PSPACE. We start with the following theorem of Canny [1].

Theorem 2 *Any sentence in the existential theory of the reals can be decided in PSPACE.*

A sentence in the first order theory of the reals is a formula of the form :

$$\exists x_1 \exists x_2 ... \exists x_n P(x_1, x_2, ..., x_n)$$

where the $x_i's$ are variables ranging over the real numbers and where $P(x_1, x_2, ..., x_n)$ is a predicate built up from $\neg$, $\wedge$, $\vee$, $=$, $<$, $>$, $+$, $\times$, 0, 1 and -1 in the usual way.

Theorem 3 *The recognition problem for point visibility graphs lies in PSPACE.*

Proof : Given a graph $G(V, E)$, we construct a formula in the existential

theory of the reals polynomial in size of G which is true if and only if G is a point visibility graph.

Suppose $(v_i, v_j) \notin E$. This means that if G admits a visibility embedding, then there must be a blocker (say, p_k) on the segment joining p_i and p_j. Let the coordinates of the points p_i, p_j and p_k be (x_i, y_i), (x_j, y_j) and (x_k, y_k) respectively. So we have :

$$\exists t_{\in}\mathbb{R}\Big((0<t)\wedge(t<1)\wedge((x_k-x_i)=t\times(x_j-x_i))\wedge((y_k-y_i)=t\times(y_j-y_i))\Big)$$

Now suppose $(v_i, v_j) \in E$. This means that if G admits a visibility embedding, no point in P lies on the segment connecting p_i and p_j to ensure visibility. So, (i) either p_k forms a triangle with p_i and p_j or (ii) p_k lies on the line passing through p_i and p_j but not between p_i and p_j. Determinants of non-collinear points is non-zero. So we have :

$$\exists t \in \mathbb{R}\Big(\big(det(x_i, x_j, x_k, y_i, y_j, y_k) > 0\big) \vee \big(det(x_i, x_j, x_k, y_i, y_j, y_k) < 0\big)\Big) \bigvee \Big(\big(t > 1\big) \vee \big(t < -1\big) \wedge \big((x_k - x_i) = t \times (x_j - x_i)\big) \wedge \big((y_k - y_i) = t \times (y_j - y_i)\big)\Big)$$

For each triple (v_i, v_j, v_k) of vertices in V, we add a $t = t_{i,j,k}$ to the existential part of the formula and the corresponding portion to the predicate. So the formula becomes:

$$\exists x_1 \exists y_1 ... \exists x_n \exists y_n \exists t_{1,2,3} .... \exists t_{n-2,n-1,n} \ P(x_1, y_1, ..., x_n, y_n, t_{1,2,3}, ..., t_{n-2,n-1,n})$$

which is of size $O(n^3)$. This proves our theorem. □

4 Planar point visibility graphs

In this section, we present a characterization, recognition and reconstruction of planar point visibility graphs. Let G be a given planar graph. If G has no CSP of vertices 3 or more, then G does not have any invisible pair, and therefore, G is a complete graph. So, G can have at most four vertices. In the same way, we derive upper bounds on the number of vertices in G based on the size of CSPs in G as follows. Let k-CSP denote a CSP of k vertices in G. Analogously k-GSP is defined.

Lemma 10 *Assume that G admits a visibility embedding. If G has at least one 3-CSP but no 4-CSP, then G has at most nine vertices.*

Proof : Consider a visibility embedding of G. Since there is no 4-GSP in

the embedding, the degree of any vertex is at least $\lceil \frac{n-1}{2} \rceil$ by Lemma 7. So, the number of edges in G is at least $\frac{n}{2} \times \lceil \frac{n-1}{2} \rceil$ which is more than $3n - 6$ for $n \geq 10$ contradicting Euler's theorem for planar graphs. So the number of vertices in G cannot exceed nine. $\square$

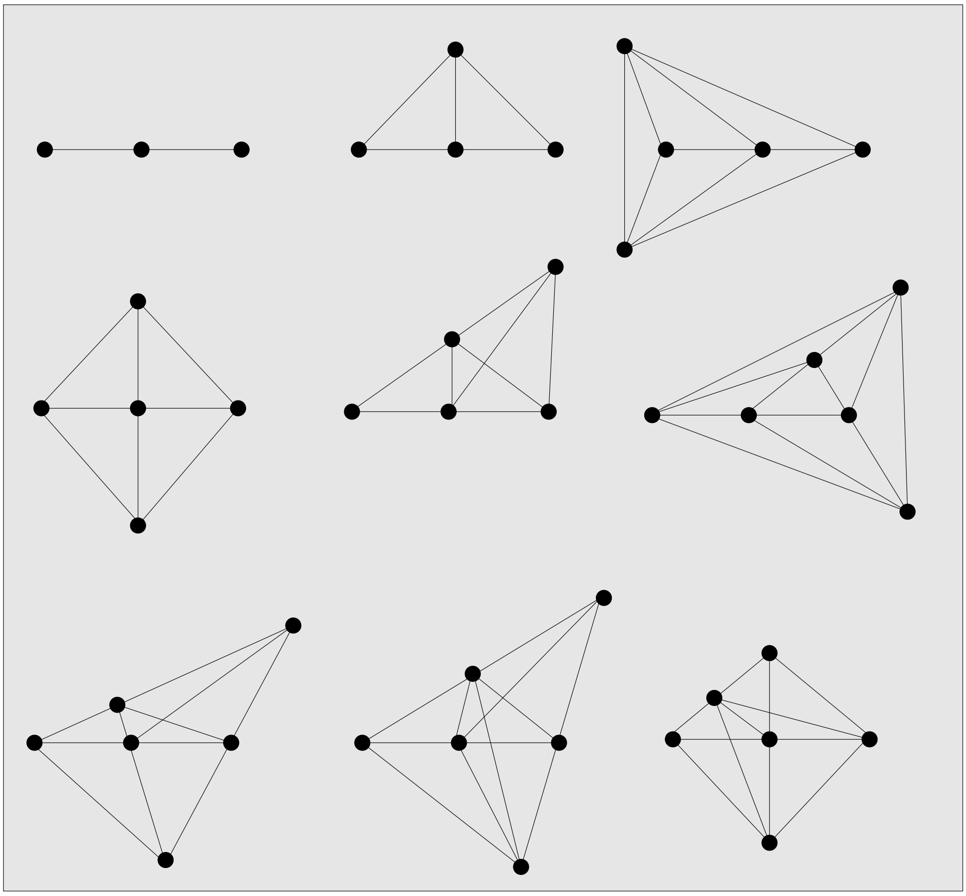

Figure 7: There are only nine planar PVGs with at least one 3-CSP but no 4-CSP. With this condition, no 7 or 8 vertices planar PVG exists.

Corollary 3 *There are nine distinct planar graphs G having at least one 3-CSP but no 4-CSP (Figure 7).*

Lemma 11 *Assume that G admits a visibility embedding. If G has at least one k-CSP for $k \geq 4$, then the number of vertices in G is at most*

$$k + \left\lfloor \frac{2k-5}{k-3} \right\rfloor$$

Proof: Consider a k-CSP in G. Since there are $n-k$ vertices outside the k-CSP, and each vertex has degree at least k by Lemma 7, we have the following inequality on the number of permissible edges of G.

$$\begin{aligned}
(k-1)+(n-k)k &\leq 3(n)-6 \\
\Rightarrow (k-1)+(n-k)k &\leq 3(k+n-k)-6 \\
\Rightarrow (k-1)+(n-k)k &\leq 3k+3(n-k)-6 \\
\Rightarrow (n-k)(k-3) &\leq 2k-5 \\
\Rightarrow (n-k) &\leq \frac{2k-5}{k-3}
\end{aligned} \tag{1}$$

Since $(n-k)$ must be an integer, we have

$$\begin{aligned}
(n-k) &\leq \left\lfloor \frac{2k-5}{k-3} \right\rfloor \\
\Rightarrow n &\leq k+\left\lfloor \frac{2k-5}{k-3} \right\rfloor
\end{aligned} \tag{2}$$

□

Corollary 4 *There are six families of planar graphs G having at least one k-CSP for $k \geq 4$ (Figures 8 and 9).*

Theorem 4 *Planar point visibility graphs can be characterized by the following graphs.*

1. *Cliques of size at most four vertices.*
2. *Nine graphs with at least a 3-CSP but no 4-CSP.*
3. *Six infinite families of graphs with a CSP of size four or more.*

Lemma 12 *Graphs given in Theorem 4 can be drawn with small integer coordinates with size $O(logn)$ bits.*

Theorem 5 *Planar point visibility graphs can be recognized and reconstructed in $O(n)$ time.*

Proof: The proof follows Theorem 4 for testing a given planar graph G for PVG.

1. If the entire G is a path or a clique having at most four vertices, then G is a PVG.

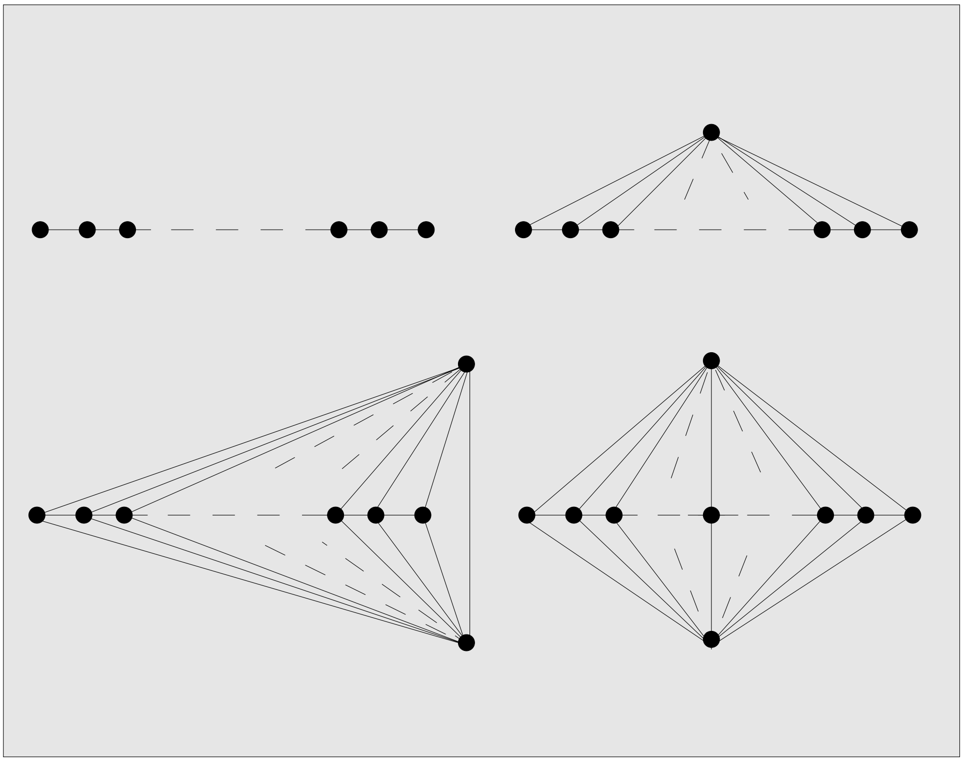

Figure 8: These four infinite families admit planar visibility embedding (given by Eppstein).

2. If G has at most six vertices, the nine graphs given in Theorem 4 having at least one 3-CSP but no 4-CSP can be tested in O(1) time using brute force method.
3. If G consists of a path of $n-1$ vertices and a vertex adjacent to all vertices of the path, then G is a PVG.
4. If G consists of a path of $n-2$ vertices and two vertices adjacent to all vertices of the path, then G is a PVG.
5. If G consists of a path of $n-2$ vertices and two vertices of degree $n-1$ and $n-2$ respectively, then G is a PVG.

If G is found to be a PVG at any stage in the above method of checking, then G can be drawn according to Lemma 12. It can be seen that testing G can be done in $O(n)$ time. □

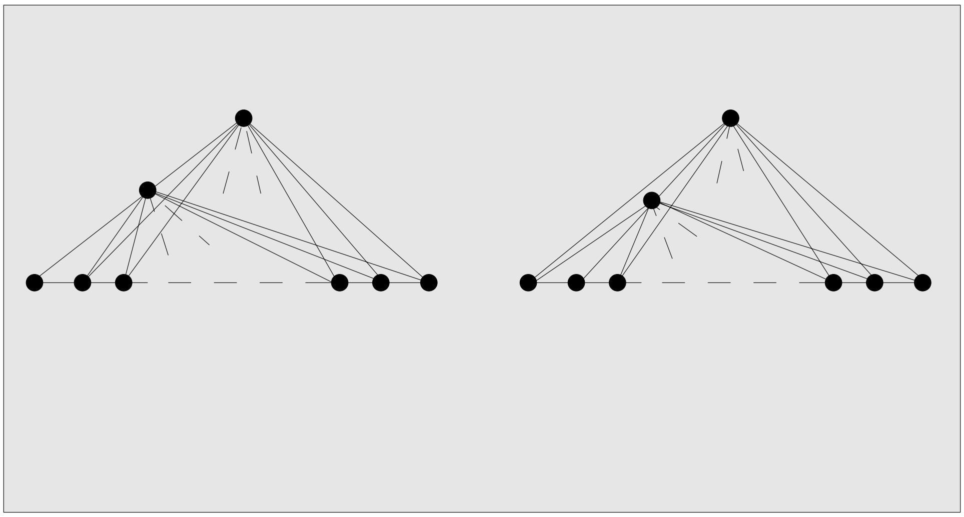

Figure 9: These two infinite families do not admit planar visibility embedding.

5 Concluding remarks

We have given two necessary conditions for recognizing point visibility graphs and conjectured that they are sufficient. Though the first necessary condition can be tested in $O(n^3)$ time, it is not clear how vertex-blockers can be assigned to every invisible pair in G in polynomial time satisfying the second necessary condition. Observe that these assignments in a visibility embedding give the ordering of collinear points along any ray starting from any point through its visible points. These rays together form an arrangement of rays in the plane. It is open whether such an arrangement can be constructed satisfying assigned vertex-blockers.

Let us consider the complexity issues of the problems of Vertex Cover, Independent Set and Maximum Clique in a point visibility graph. Let G be a graph of n vertices, not necessarily a PVG. We construct another graph G' such that (i) G is an induced subgraph of G', and (ii) G' is a PVG. Let C be a convex polygon drawn along with all its diagonals, where every vertex v_i of G corresponds to a vertex p_i of C. For every edge $(v_i, v_j) \notin G$, introduce a blocker p_t on the edge (p_i, p_j) such that p_t is visible to all points of C and all blockers added so far. Add edges from p_t to all vertices of C and blockers in C. The graph corresponding to this embedding is called G'. So, G' and its embedding can be constructed in polynomial time. Let the sizes of the

minimum vertex cover, maximum independent set and maximum clique in G be k_1, k_2 and k_3 respectively. If x is the number of blockers addded to C, then the sizes of the minimum vertex cover, maximum independent set and maximum clique in G' are $k_1 + x$, k_2 and $k_3 + x$ respectively. Hence, the problems remain NP-Hard.

Theorem 6 *The problems of Vertex Cover, Independent Set and Maximum Clique remain NP-hard on point visibility graphs.*

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